A. BASIC ARITHMETIC CONCEPTS

For any number *a*, exactly one of the following is true:

• ***a*** **is negative**	• $\boldsymbol{a = 0}$	• ***a*** **is positive**

The only number that is equal to its opposite is 0.

Column A	Column B

Example 1.

$$a - b = -(a - b)$$

Column A	Column B
a	b

Example 2.

What is the sum of the product and quotient of 8 and 8?

(A) 16 (B) 17 (C) 63 (D) 64 (E) 65

- **The product of 0 and any number is 0. For any number *a*: $a \times 0 = 0$.**
- **Conversely, if the product of two numbers is 0, *at least one* of them must be 0:**

$$ab = 0 \Rightarrow a = 0 \text{ or } b = 0.$$

- **The product of an *even* number of negative factors is positive.**
- **The product of an *odd* number of negative factors is negative.**

- **The *reciprocal* of any nonzero number *a* is $\frac{1}{a}$.**
- **The product of any number and its reciprocal is 1:**

$$a \times \left(\frac{1}{a}\right) = 1.$$

The sum of any number and its opposite is 0:

$$a + (-a) = 0.$$

Column A	Column B

Example 3.

Column A	Column B
The product of the integers from –7 to 2	The product of the integers from –2 to 7

Example 4.

Column A	Column B
(–1)(2)(–3)(4)(–5)	(1)(–2)(3)(–4)(5)

The *integers* are $\{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$.

The *positive integers* are $\{1, 2, 3, 4, 5, \dots\}$.

The *negative integers* are $\{\dots, -5, -4, -3, -2, -1\}$.

NOTE: The integer 0 is neither positive nor negative. Therefore, if a question on the GRE asks how many positive numbers have a certain property, and the only numbers with that property are –2, –1, 0, 1, and 2, the answer is **2**.

Consecutive integers are two or more integers written in sequence in which each integer is 1 more than the preceding integer. For example:

22, 23   6, 7, 8, 9   –2, –1, 0, 1 $n, n + 1, n + 2, n + 3$

Example 5.

If the sum of three consecutive integers is less than 75, what is the greatest possible value of the smallest one?

(A) 23   (B) 24   (C) 25   (D) 26   (E) 27

Example 6.

If $2 < x < 4$ and $3 < y < 7$, what is the largest integer value of $x + y$?

(A) 7   (B) 8   (C) 9   (D) 10   (E) 11

Example 7.

How many positive integers less than 100 have a remainder of 3 when divided by 7?

The tables below summarize three important facts:

1. **If two integers are both even or both odd, their sum and difference are even.**
2. **If one integer is even and the other odd, their sum and difference are odd.**
3. **The product of two integers is even unless both of them are odd.**

+ and –	even	odd
even	even	odd
odd	odd	even

×	even	odd
even	even	even
odd	even	odd

Prime numbers

Prime numbers are numbers that have exactly **two** divisors: 1 and themselves.

2, 3, 5, 7, 11, 13, 17, 19, 23, …

Remarks:

1 is NOT a prime number.

2 is the ONLY even prime number.

A number is the square of a prime number if and only if it has three divisors.

Multiples

$M_2 = \{2, 4, 6,8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 10, 42, \ldots\}$

$M_3 = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45 \ldots\}$

Lowest Common Multiple $LCM(2, 3) = 6$

$LCM(540, 1134)?$

540	5
108	2
54	2
27	3
9	3
3	3
1	

1134	2
567	7
81	3
27	3
9	3
3	3
1	

Prime factorization

$$540 = 2^2 \times 3^3 \times 5$$

$$1134 = 2 \times 3^4 \times 7$$

$$LCM(540, 1134) = 2^2 \times 3^4 \times 5 \times 7 = 11340$$

Find the LCM of:

a) 90 and 140

b) 14 and 15

c) $a = 2^2 \times 3^5 \times 11$
$b = 2^4 \times 3^2$

Remarks:

The LCM of two numbers that do not have any common prime factors is their product.

A number divisible by 2 numbers a and b is a multiple of their LCM.

Factors / Divisors

$F_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$F_{48} = \{1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}$

Greatest common factor $GCF(36, 48) = 12$

36	2
18	2
9	3
3	3
1	

48	2
24	2
12	3
4	2
2	2
1	

$36 = 2^2 \times 3^2$

$48 = 2^4 \times 3$

$GCF(36, 48) = 2^2 \times 3 = 12$

Find the GCF of:

a) 90 and 140

b) 14 and 15

c) $a = 2^2 \times 3^5 \times 11$
$b = 2^4 \times 3^2$

Remarks

The GCF of two numbers who do not have any common prime factors is 1.

$LCM(a,b) \times GCF(a,b) = a \times b$

a is a divisor of $b \Leftrightarrow b$ is a multiple of a

$a = 2^2 \times 3$

The prime factors of a are 2 and 3

The factors of a are $1, 2, 2^2, 3, 2 \times 3, 2^2 \times 3$

The multiples of a are $2^2 \times 3, 2^3 \times 3, 2^2 \times 3^2, 2^2 \times 3 \times 5, \ldots$

Exponents

For any numbers b and c and positive integers m and n:

(i) $b^m b^n = b^{m+n}$ (ii) $\frac{b^m}{b^n} = b^{m-n}$ (iii) $(b^m)^n = b^{mn}$

(iv) $b^m c^m = (bc)^m$

For any positive integer n:

- **$0^n = 0$**
- **if a is positive, then a^n is positive**
- **if a is negative and n is even, then a^n is positive**
- **if a is negative and n is odd, then a^n is negative.**

Example 10.

If $2^x = 32$, what is x^2?

(A) 5 (B) 10 (C) 25 (D) 100 (E) 1024

Example 11.

If $3^a \times 3^b = 3^{100}$, what is the average (arithmetic mean) of a and b?

Example 12.

Column A	Column B
$(-13)^{10}$	$(-13)^{25}$

Square roots

For any positive number ***a***, there is a positive number ***b*** that satisfies the equation $b^2 = a$. That number is called the square root of ***a*** and we write $b = \sqrt{a}$.

So, for any positive number ***a***: $(\sqrt{a})^2 = \sqrt{a} \times \sqrt{a} = a$.

For any positive numbers *a* and *b*:

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

CAUTION: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. For example:

$$5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

CAUTION: Although it is always true that $(\sqrt{a})^2 = a$, $\sqrt{a^2} = a$ is true *only if a* is positive:

$$\sqrt{(-5)^2} = \sqrt{25} = 5, \textit{ not } -5.$$

Example 13.

What is the circumference of a circle whose area is 10π?

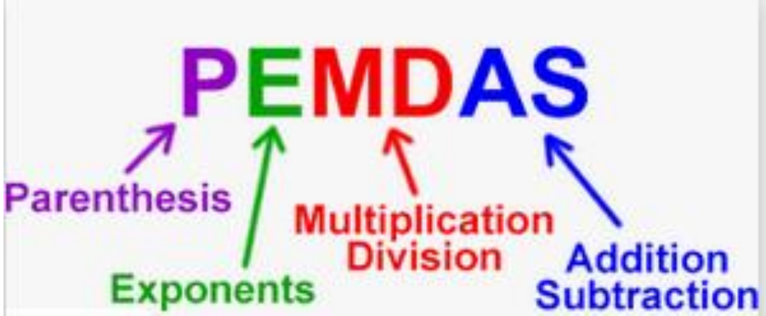
(A) 5π (B) 10π (C) $\pi\sqrt{10}$ (D) $2\pi\sqrt{10}$
(E) $\pi\sqrt{20}$

Column A	Column B

Example 14.

$\sqrt{x^{20}}$	$(x^5)^2$

Order of operations



Column A	Column B
Example 15.	
$5(a-7)$	$5a-7$

Column A	Column B
Example 16.	
$\frac{50+x}{5}$	$10+x$

Example 17.

If $a = 9 \times 8321$ and $b = 9 \times 7321$, what is the value of $a - b$?

·B. FRACTIONS AND DECIMALS

Outer → Inner →

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Column A	Column B
Example 1.	
.2139	.239

Example 2.
Which of the following lists the fractions $\frac{2}{3}$, $\frac{5}{8}$, and $\frac{13}{20}$ in order from least to greatest?

(A) $\frac{2}{3}, \frac{5}{8}, \frac{13}{20}$ (B) $\frac{5}{8}, \frac{2}{3}, \frac{13}{20}$

(C) $\frac{5}{8}, \frac{13}{20}, \frac{2}{3}$ (D) $\frac{13}{20}, \frac{5}{8}, \frac{2}{3}$

(E) $\frac{13}{20}, \frac{2}{3}, \frac{5}{8}$

Column A	Column B

Example 3.

$$0 < x < y$$

Column A	Column B
$\frac{1}{x} - \frac{1}{y}$	0

Example 4.

For any positive integer n: $n!$ means the product of all the integers from 1 to n. What is the value of $\frac{6!}{8!}$?

(A) $\frac{1}{56}$ (B) $\frac{1}{48}$ (C) $\frac{1}{8}$ (D) $\frac{1}{4}$ (E) $\frac{3}{4}$

Column A	Column B

Example 5.

Column A	Column B
3.75×10^4	$37{,}500{,}000 \div 10^3$

Example 6.

Express the product, $\frac{3}{4} \times \frac{8}{9} \times \frac{15}{16}$, in lowest terms.

Example 7.

If $\frac{4}{7}$ of the 350 sophomores at Monroe High School are girls, and $\frac{7}{8}$ of them play on a team, how many sophomore girls do <u>not</u> play on a team?

Example 8.

In the meat department of a supermarket, 100 pounds of chopped meat was divided into packages, each of which weighed $\frac{4}{7}$ of a pound. How many packages were there?

Example 9.

In a jar, $\frac{1}{2}$ of the marbles are red, $\frac{1}{4}$ are white, and $\frac{1}{5}$ are blue. What fraction of the marbles are neither red, white, nor blue?

Example 10.

Lindsay ate $\frac{1}{3}$ of a cake and Emily ate $\frac{1}{4}$ of it. What fraction of the cake was still uneaten?

Example 11.

Lindsay ate $\frac{1}{3}$ of a cake and Emily ate $\frac{1}{4}$ of what was left. What fraction of the cake was still uneaten?

Mixed Numbers

$$2\frac{2}{5} = 2 + \frac{2}{5} = \frac{2}{1} + \frac{2}{5} = \frac{2\times 5}{1\times 5} + \frac{2}{5} = \frac{12}{5}$$

Mixed number

Improper fraction

$$\frac{14}{5} = \frac{10+4}{5} = \frac{10}{5} + \frac{4}{5} = 2 + \frac{4}{5} = 2\frac{4}{5}$$

Improper fraction

Mixed number

Percentages

$$2\% = \frac{2}{100} = 0.02$$

$$20\% = \frac{20}{100} = 0.2$$

$$120\% = \frac{120}{100} = 1.2$$

Find 15% of 120

What percent of 120 is 40?

30 is 40% of what number?

Example 1.

Charlie gave 20% of his baseball cards to Kenne and 15% to Paulie. If he still had 520 cards, how many did he have originally?

(A) 555 (B) 700 (C) 800 (D) 888 (E) 1000

Example 2.

After Ruth gave 110 baseball cards to Alison and 75 to Susanna, she still had 315 left. What percent of her cards did Ruth give away?

(A) 25% (B) $33\frac{1}{3}\%$ (C) 37% (D) 40% (E) 50%

Example 3.

In 1985 the populations of town A and town B were the same. From 1985 to 1995 the population of town A increased by 60% while the population of town B decreased by 60%. In 1995, the population of town B was what percent of the population of town A?

(A) 25% (B) 36% (C) 40% (D) 60% (E) 120%

Finding an amount after increasing or decreasing by $k\%$

- **To increase a number by $k\%$, multiply it by $(1 + k\%)$.**
- **To decrease a number by $k\%$, multiply it by $(1 - k\%)$.**

Example 4.

Store B always sells CDs at 60% off the list price. Store A sells its CDs at 40% off the list price, but often runs a special sale during which it reduces its prices by 20%.

<u>Column A</u>	<u>Column B</u>
The price of a CD when it is on sale at store A	The price of the same CD at store B

Finding an initial amount before increasing or decreasing by $k\%$

- **If a number is the result of increasing another number by $k\%$, to find the original number, divide by $(1 + k\%)$.**
- **If a number is the result of decreasing another number by $k\%$, to find the original number, divide it by $(1 - k\%)$.**

Finding the percent of change (increase or decrease)

- The ***percent increase*** of a quantity is

$$\frac{\text{actual increase}}{\text{original amount}} \times 100\%.$$

- The ***percent decrease*** of a quantity is

$$\frac{\text{actual decrease}}{\text{original amount}} \times 100\%.$$

- **A decrease of *a*% followed by a decrease of *b*% *always* results in a smaller decrease than a single decrease of (*a* + *b*)%.**
- **An increase of *a*% followed by an increase of *b*% *always* results in a larger increase than a single increase of (*a* + *b*)%.**
- **An increase (or decrease) of *a*% followed by another increase (or decrease) of *a*% is *never* the same as a single increase (or decrease) of 2*a*%.**

Column A	Column B

Example 5.

Sally and Heidi were both hired in January at the same salary. Sally got two 40% raises, one in July and another in November. Heidi got one 90% raise in October.

Column A	Column B
Sally's salary at the end of the year	Heidi's salary at the end of the year

Example 6.

In January, the value of a stock increased by 25%, and in February, it decreased by 20%. How did the value of the stock at the end of February compare with its value at the beginning of January?

(A) It was less.
(B) It was the same.
(C) It was 5% greater.
(D) It was more than 5% greater.
(E) It cannot be determined from the information given.

Example 7.

From 1989 to 1990, the number of applicants to a college increased 15% to 5060. How many applicants were there in 1989?

(A) 759 (B) 4301 (C) 4400 (D) 5819 (E) 5953

Example 8.

The population of a country doubled every 10 years from 1960 to 1990. What was the percent increase in population during this time?

(A) 200% (B) 300% (C) 700% (D) 800% (E) 1000%

D. RATIOS AND PROPORTIONS

Example

The ratio of girls to boys in a class is 3 to 8 → if the number of girls is $3x$, the number of boys would be $8x$ and the ratio of boys to students would be $\frac{8x}{3x + 8x} = \frac{8}{11}$

Example 1.

Last year, the ratio of the number of tennis matches that Central College's women's team won to the number of matches they lost was 7:3. What percent of their matches did the team win?

Example 2.

If 45% of the students at a college are male, what is the ratio of male students to female students?

Example 3.

If the ratio of men to women in a particular dormitory is 5:3, which of the following could not be the number of residents in the dormitory?

(A) 24 (B) 40 (C) 96 (D) 150 (E) 224

Example 4.

The measures of the two acute angles in a right triangle are in the ratio of 5:13. What is the measure of the larger angle?

(A) 25° (B) 45° (C) 60° (D) 65° (E) 75°

Example 5.

The concession stand at Cinema City sells popcorn in three sizes: large, super, and jumbo. One day, Cinema City sold 240 bags of popcorn, and the ratio of large to super to jumbo was 8:17:15. How many super bags of popcorn were sold that day?

(A) 48 (B) 90 (C) 102 (D) 108 (E) 120

Example 6.

If the ratio of large to super to jumbo bags of popcorn sold at Cinema City was 8:17:15, what percent of the bags sold were super?

(A) 20% (B) 25% (C) $33\frac{1}{3}$% (D) 37.5%

(E) 42.5%

Column A	Column B

Example 7.

Jar A and jar B each have 70 marbles,
all of which are red, white, or blue.
In jar A, R:W = 2:3 and W:B = 3:5.
In jar B, R:W = 2:3 and W:B = 4:5.

The number of white marbles in jar A	The number of white marbles in jar B

Example 8.

If $\frac{3}{7} = \frac{x}{84}$, what is the value of x?

(A) 12 (B) 24 (C) 36 (D) 42 (E) 48

Example 9.

If $\frac{x+2}{17} = \frac{x}{16}$, what is the value of $\frac{x+6}{19}$?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) 3

x and y are proportional $\Leftrightarrow \frac{x}{y} = k$

x and y are inversely proportional $\Leftrightarrow x.y = k$

Example 10.

A state law requires that on any field trip the ratio of the number of chaperones to the number of students must be at least 1:12. If 100 students are going on a field trip, what is the minimum number of chaperones required?

(A) 6 (B) 8 (C) $8\frac{1}{3}$ (D) 9 (E) 12

Example 11.

Brigitte solved 24 math problems in 15 minutes. At this rate, how many problems can she solve in 40 minutes?

(A) 25 (B) 40 (C) 48 (D) 60 (E) 64

Example 12.

If Stefano types at the rate of 35 words per minute, how long will it take him to type 987 words?

Example 13.

If Mario types at the rate of 35 words per minute, how many words can he type in 85 minutes?

Example 14.

If 3 apples cost 50¢, how many apples can you buy for $20?

(A) 20 (B) 60 (C) 120 (D) 600 (E) 2000

Example 15.

If a apples cost c cents, how many apples can be bought for d dollars?

(A) $100acd$ (B) $\frac{100d}{ac}$ (C) $\frac{ad}{100c}$ (D) $\frac{c}{100ad}$

(E) $\frac{100ad}{c}$

Example 16.

A hospital needs 150 pills to treat 6 patients for a week. How many pills does it need to treat 10 patients for a week?

Example 17.

A hospital has enough pills on hand to treat 10 patients for 14 days. How long will the pills last if there are 35 patients?

Example 18.

If 15 workers can pave a certain number of driveways in 24 days, how many days will 40 workers take, working at the same rate, to do the same job?

(A) 6 (B) 9 (C) 15 (D) 24 (E) 40

Example 19.

If 15 workers can pave 18 driveways in 24 days, how many days would it take 40 workers to pave 22 driveways?

(A) 6 (B) 9 (C) 11 (D) 15 (E) 18

E. AVERAGES

The ***average*** of a set of *n* numbers is the sum of those numbers divided by *n*.

$$\text{average} = \frac{\text{sum of the } n \text{ numbers}}{n}$$

If the weights of three children are 80, 90, and 76 pounds, respectively, to calculate the average weight of the children, you would add the three weights and divide by 3:

$$\frac{80+90+76}{3} = \frac{246}{3} = 82$$

Example 1.

One day a supermarket received a delivery of 25 frozen turkeys. If the average (arithmetic mean) weight of a turkey was 14.2 pounds, what was the total weight, in pounds, of all the turkeys?

Example 2.

Sheila took five chemistry tests during the semester and the average (arithmetic mean) of her test scores was 85. If her average after the first three tests was 83, what was the average of her fourth and fifth tests?

(A) 83 (B) 85 (C) 87 (D) 88 (E) 90

Example 3.

If the average (arithmetic mean) of 25, 31, and x is 37, what is the value of x?

(A) 31 (B) 37 (C) 43 (D) 55 (E) 56

Example 4.

Column A	Column B
The average (arithmetic mean) of the integers from 0 to 12	The average (arithmetic mean) of the integers from 1 to 12

Whenever *n* numbers form an arithmetic sequence (one in which the difference between any two consecutive terms is the same): (i) if *n* is odd, the average of the numbers is the middle term in the sequence and (ii) if *n* is even, the average of the numbers is the average of the two middle terms.

For example, in the arithmetic sequence 6, 9, 12, 15, 18, the average is the middle number, **12**; in the sequence 10, 20, 30, 40, 50, 60, the average is **35**, the average of the two middle numbers—30 and 40.

To calculate the weighted average of a set of numbers, multiply each number in the set by the number of times it appears, add all the products, and divide by the total number of numbers in the set.

Example 5.

On Thursday, 20 of the 25 students in a chemistry class took a test and their average was 80. On Friday, the other 5 students took the test, and their average was 90. What was the average (arithmetic mean) for the entire class?

(A) 80 (B) 82 (C) 84 (D) 85 (E) 88

$$\boldsymbol{Average\ speed} = \frac{\boldsymbol{total\ distance}}{\boldsymbol{total\ time}}$$

Example 6.

For the first 3 hours of his trip, Justin drove at 50 miles per hour. Then, due to construction delays, he drove at only 40 miles per hour for the next 2 hours. What was his average speed, in miles per hour, for the entire trip?

(A) 40 (B) 43 (C) 46 (D) 48 (E) 50

Example 6a.

For the first 100 miles of his trip, Justin drove at 50 miles per hour, and then due to construction delays, he drove at only 40 miles per hour for the next 120 miles. What was his average speed, in miles per hour, for the entire trip?

Example 7.

During a 10-day period, Jorge received the following number of phone calls each day: 2, 3, 9, 3, 5, 7, 7, 10, 7, 6. What is the average (arithmetic mean) of the median and mode of this set of data?

(A) 6 (B) 6.25 (C) 6.5 (D) 6.75 (E) 7

F. POLYNOMIALS

Example 1.

What is the value of $-3a^2b$ when $a = -4$ and $b = 0.5$?

(A) -72 (B) -24 (C) 24 (D) 48 (E) 72

Example 2.

What is the sum of $5x^2 + 10x - 7$ and $3x^2 - 4x + 2$?

Example 3.

Subtract $3x^2 - 4x + 2$ from $5x^2 + 10x - 7$.

Example 4.

What is the average (arithmetic mean) of $5x^2 + 10x - 7$, $3x^2 - 4x + 2$, and $4x^2 + 2$?

Example 5.

What is the product of $3xy^2z^3$ and $-2x^2y^2$?

Example 6.

What is the product of $2a$ and $3a^2 - 6ab + b^2$?

Example 7.

What is the value of $(x - 2)(x + 3) - (x - 4)(x + 5)$?

Example 8.

If $a - b = 7$ and $a + b = 13$, what is the value of $a^2 - b^2$?

(A) –120 (B) 20 (C) 91 (D) 120 (E) 218

Example 9.

If $x^2 + y^2 = 36$ and $(x + y)^2 = 64$, what is the value of xy?

(A) 14 (B) 28 (C) 100 (D) 128 (E) 2304

Example 10.

What is the quotient when $32a^2b + 12ab^3c$ is divided by $8ab$?

Column A	Column B

Example 11.

Column A	Column B
The value of $x^2 + 4x + 4$ when $x = 95.9$	The value of $x^2 - 4x + 4$ when $x = 99.5$

Example 12.

What is the value of $(1{,}000{,}001)^2 - (999{,}999)^2$?

Example 13.

What is the sum of the reciprocals of x^2 and y^2?

Example 14.

What is the value of $\dfrac{4x^3 - x}{(2x+1)(6x-3)}$ when $x = 9999$?

G. SOLVING EQUATIONS AND INEQUALITIES

Example 1.

If $\frac{1}{2}x + 3(x - 2) = 2(x + 1) + 1$, what is the value of x?

Example 2.

For what real number n is it true that $3(n - 20) = n$?

(A) –10 (B) 0 (C) 10 (D) 20 (E) 30

Example 3.

Three brothers divided a prize as follows. The oldest received $\frac{2}{5}$ of it, the middle brother received $\frac{1}{3}$ of it, and the youngest received the remaining $120. What was the value of the prize?

Example 4.

If $a = 3b - c$, what is the value of b in terms of a and c?

Example 5.

If $x - 4 = 11$, what is the value of $x - 8$?

(A) –15 (B) –7 (C) –1 (D) 7 (E) 15

Example 6.

If $2x - 5 = 98$, what is the value of $2x + 5$?

Example 7.

If w is an integer, and the average (arithmetic mean) of 3, 4, and w is less than 10, what is the greatest possible value of w ?

(A) 9 (B) 10 (C) 17 (D) 22 (E) 23

Example 8.

For what value of x is $\frac{4}{x} + \frac{3}{5} = \frac{10}{x}$?

(A) 5 (B) 10 (C) 20 (D) 30 (E) 50

Example 9. If x is positive, and $y = 5x^2 + 3$, which of the following is an expression for x in terms of y?

(A) $\sqrt{\frac{y}{5} - 3}$ (B) $\sqrt{\frac{y-3}{5}}$ (C) $\frac{\sqrt{y-3}}{5}$

(D) $\frac{\sqrt{y} - 3}{5}$ (E) $\frac{\sqrt{y} - \sqrt{3}}{5}$

Example 10.

If $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, what is a in terms of b and c?

Example 11.

If $a > 0$ and $a^2 + b^2 = c^2$, what is a in terms of b and c?

Example 12.

If x is a positive number and $x^2 + 64 = 100$, what is the value of x?

(A) 6 (B) 12 (C) 13 (D) 14 (E) 36

Example 13.

What is the largest value of x that satisfies the equation $2x^2 - 3x = 0$?

(A) 0 (B) 1.5 (C) 2 (D) 2.5 (E) 3

Example 14.

If $2^{x+3} = 32$, what is the value of 3^{x+2}?

(A) 5 (B) 9 (C) 27 (D) 81 (E) 125

Example 15. If $4^{w+3} = 8^{w-1}$, what is the value of w?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 9

Column A	Column B

Example 16.

$$x + y = 10$$
$$x - y = 2$$

x	y

Example 17.

If $3a + 5b = 10$ and $5a + 3b = 30$, what is the average (arithmetic mean) of a and b?

(A) 2.5 (B) 4 (C) 5 (D) 20
(E) It cannot be determined from the information given.

Column A	Column B

Example 18.

$$7a - 3b = 200$$
$$7a + 3b = 100$$

a	b

Column A	Column B

Example 19.

$$x + y = 10$$
$$y = x - 2$$

x	y

H. WORD PROBLEMS

Example 1a.

What is 4% of 4% of 40,000?

Example 1b.

In a lottery, 4% of the tickets printed can be redeemed for prizes, and 4% of those tickets have values in excess of $100. If the state prints 40,000 tickets, how many of them can be redeemed for more than $100?

Example 2a.

If $x + 7 = 2(x - 8)$, what is the value of x ?

Example 2b.

In 7 years Erin will be twice as old as she was 8 years ago. How old is Erin now?

Example 1b.

In a lottery, 4% of the tickets printed can be redeemed for prizes, and 4% of those tickets have values in excess of $100. If the state prints 40,000 tickets, how many of them can be redeemed for more than $100?

Example 2b.

In 7 years Erin will be twice as old as she was 8 years ago. How old is Erin now?

Example 3.

In 1980, Judy was 3 times as old as Adam, but in 1984 she was only twice as old as he was. How old was Adam in 1990?

(A) 4 (B) 8 (C) 12 (D) 14 (E) 16

Example 4.

How much longer, in *seconds*, is required to drive 1 mile at 40 miles per hour than at 60 miles per hour?

Example 5.

Avi drove from his home to college at 60 miles per hour. Returning over the same route, there was a lot of traffic, and he was only able to drive at 40 miles per hour. If the return trip took 1 hour longer, how many miles did he drive each way?

(A) 2 (B) 3 (C) 5 (D) 120 (E) 240

Example 6.

Lindsay is trying to collect all the cards in a special commemorative set of baseball cards. She currently has exactly $\frac{1}{4}$ of the cards in that set. When she gets 10 more cards, she will then have $\frac{1}{3}$ of the cards. How many cards are in the set?

(A) 30 (B) 60 (C) 120 (D) 180 (E) 240

Example 7.

Jen, Ken, and Len have a total of \$390. Jen has 5 times as much as Len, and Ken has $\frac{3}{4}$ as much as Jen. How much money does Ken have?

(A) \$40 (B) \$78 (C) \$150 (D) \$195 (E) \$200

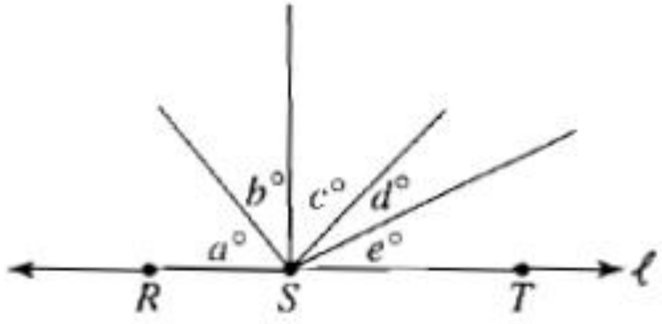
I. LINES AND ANGLES

When two lines intersect, four angles are formed. The two angles in each pair of opposite angles are called ***vertical angles***.

Vertical angles have equal measures.

Example 1.

In the figure below, R, S, and T are all on line ℓ. What is the average of a, b, c, d, and e?

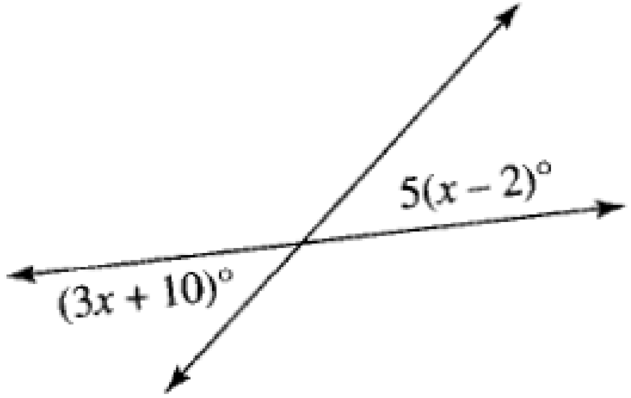


(A) 18 (B) 36 (C) 45 (D) 90 (E) 180

Example 2.

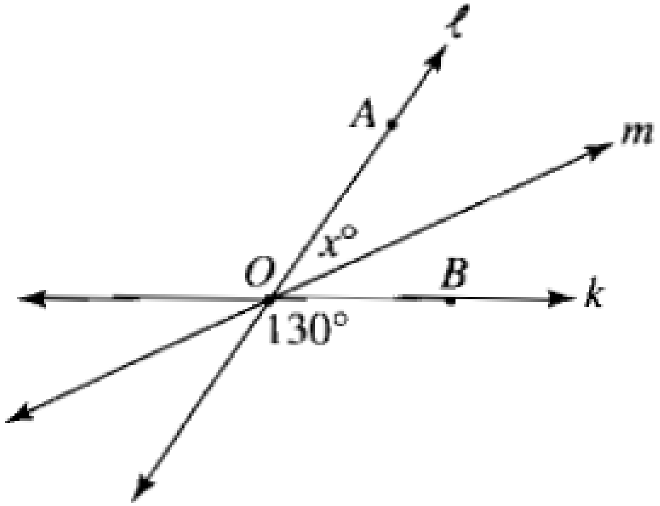
In the figure at the right, what is the value of x?

(A) 6 (B) 8
(C) 10 (D) 20
(E) 40



Example 3.

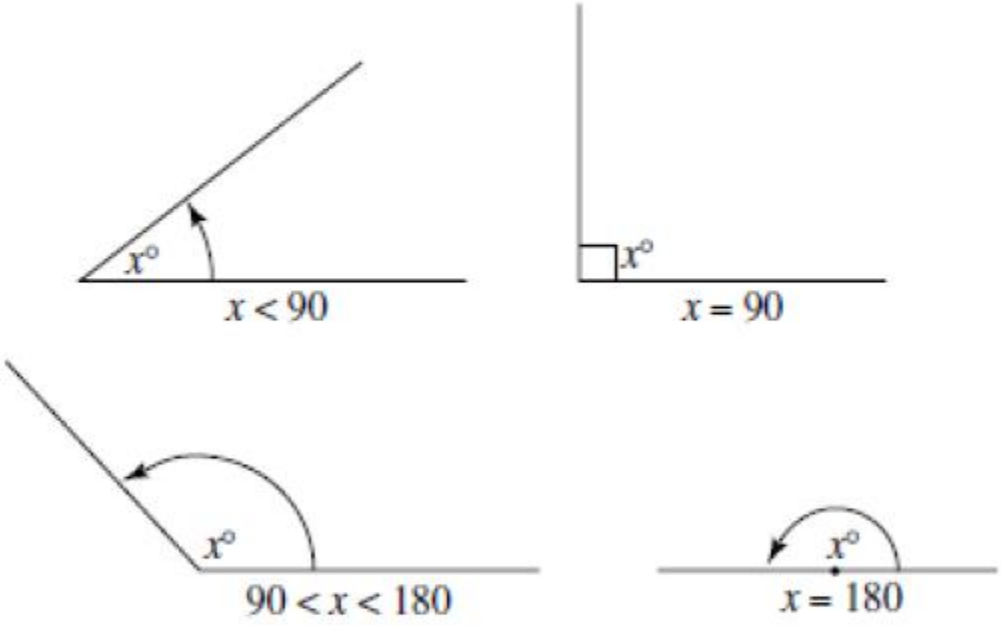
In the figure at the right, lines k, ℓ, and m intersect at O. If line m bisects $\angle AOB$, what is the value of x?



(A) 25 (B) 35 (C) 45 (D) 50 (E) 60

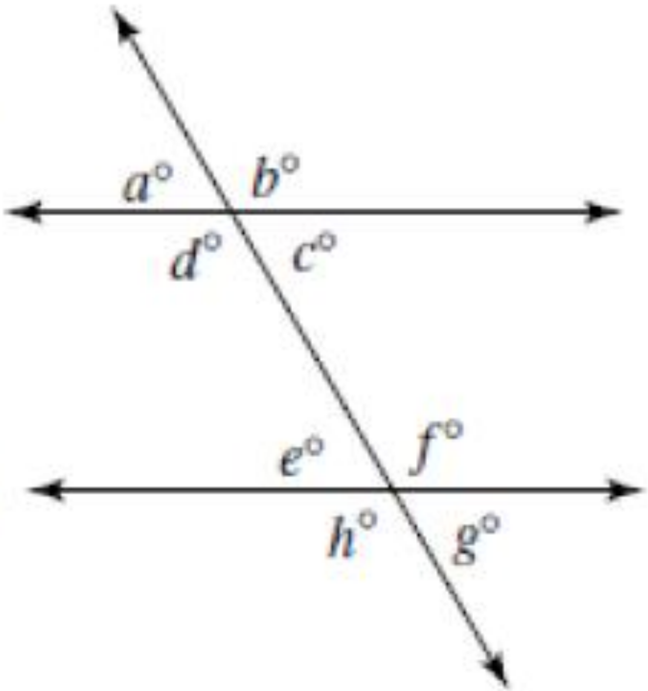
Angles are classified according to their degree measures.

- **An *acute* angle measures less than 90°.**
- **A *right* angle measures 90°.**
- **An *obtuse* angle measures more than 90° but less than 180°.**
- **A *straight* angle measures 180°.**



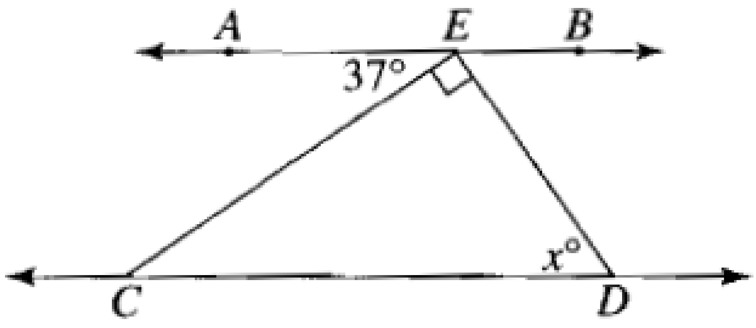
If a pair of parallel lines is cut by a transversal that is *not* perpendicular to the parallel lines:

- **Four of the angles are acute, and four are obtuse.**
- **All four acute angles are congruent: $a = c = e = g$.**
- **All four obtuse angles are congruent: $b = d = f = h$.**
- **The sum of any acute angle and any obtuse angle is 180°: for example, $d + e = 180$, $c + f = 180$, $b + g = 180$, ...**



Example 4.

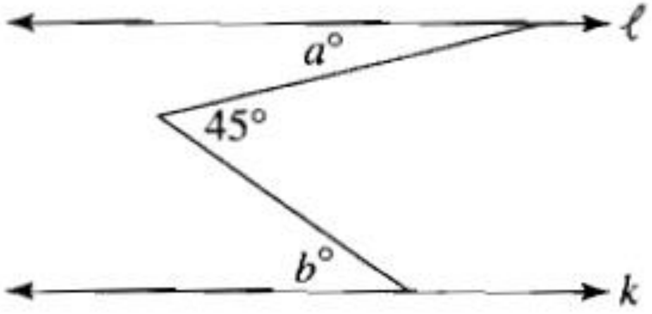
In the figure below, *AB* is parallel to *CD*. What is the value of *x*?



(A) 37 (B) 45 (C) 53 (D) 63 (E) 143

Example 5.

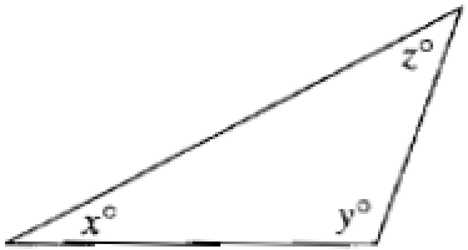
In the figure below, lines ℓ and *k* are parallel. What is the value of *a* + *b*?



(A) 45 (B) 60 (C)75 (D) 90 (E) 135

J. TRIANGLES

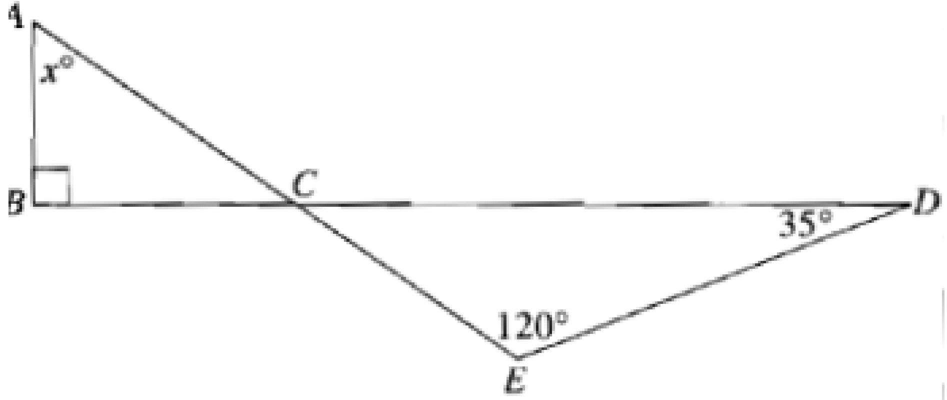
In any triangle, the sum of the measures of the three angles is 180°:
$x + y + z = 180$.



The measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles.

Example 1.

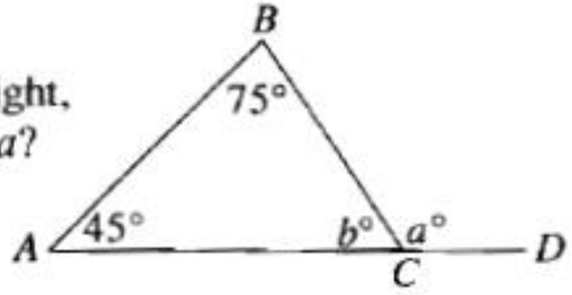
In the figure below, what is the value of x?



(A) 25 (B) 35 (C) 45 (D) 55 (E) 65

Example 2.

In the figure at the right, what is the value of a?



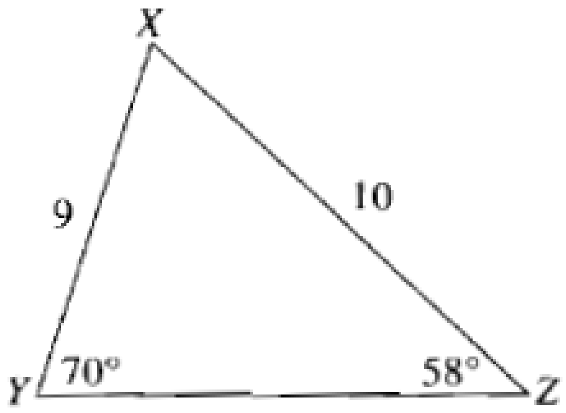
(A) 45 (B) 60 (C) 75 (D) 120 (E) 135

In any triangle:

- **the longest side is opposite the largest angle;**
- **the shortest side is opposite the smallest angle;**
- **sides with the same length are opposite angles with the same measure.**

Example 3.

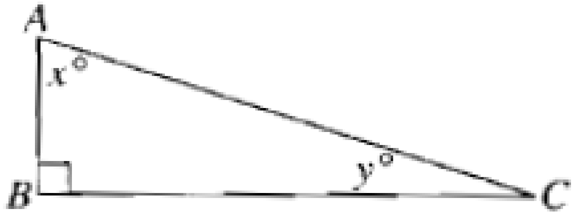
Which of the following statements concerning the length of side YZ is true?



(A) $YZ < 9$
(B) $YZ = 9$
(C) $9 < YZ < 10$
(D) $YZ = 10$
(E) $YZ > 10$

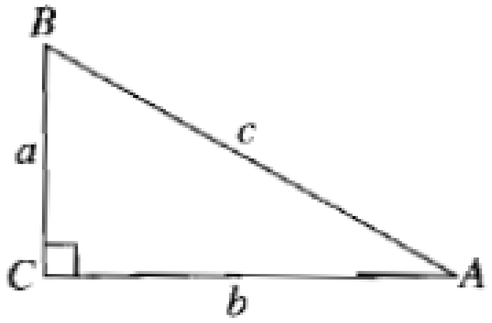
In any right triangle, the sum of the measures of the two acute angles is 90°.

Example 4.



The average of x and y	45

Let a, b, and c be the sides of $\triangle ABC$, with $a \le b \le c$. If $\triangle ABC$ is a right triangle, $a^2 + b^2 = c^2$; and if $a^2 + b^2 = c^2$, then $\triangle ABC$ is a right triangle.



Example 5.

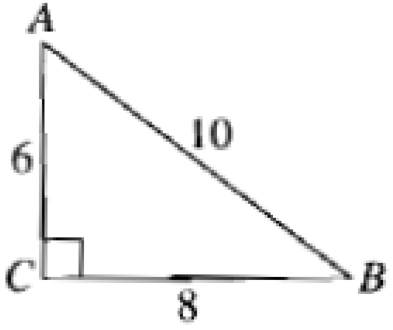
Which of the following are *not* the sides of a right triangle?

(A) 3, 4, 5 (B) 1, 1, $\sqrt{2}$ (C) 1, $\sqrt{3}$, 2

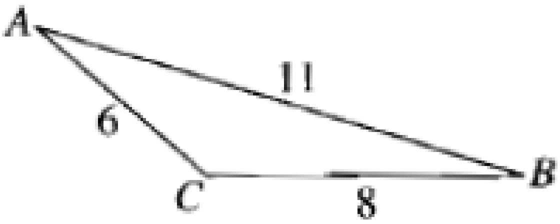
(D) $\sqrt{3}, \sqrt{4}, \sqrt{5}$ (E) 30, 40, 50

Let a, b, and c be the sides of $\triangle ABC$, with $a \le b \le c$.

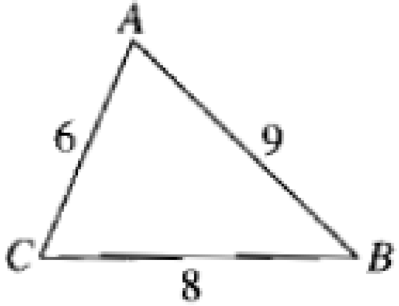
- $a^2 + b^2 = c^2$ if and only if angle C is a right angle. ($\triangle ABC$ is a right triangle.)
- $a^2 + b^2 < c^2$ if and only if angle C is obtuse. ($\triangle ABC$ is an obtuse triangle.)
- $a^2 + b^2 > c^2$ if and only if angle C is acute. ($\triangle ABC$ is an acute triangle.)



$6^2 + 8^2 = 10^2$

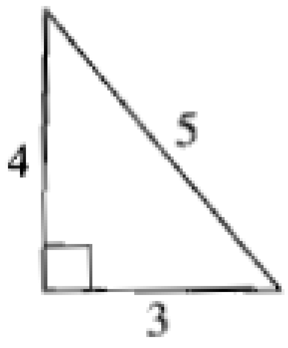


$6^2 + 8^2 < 11^2$

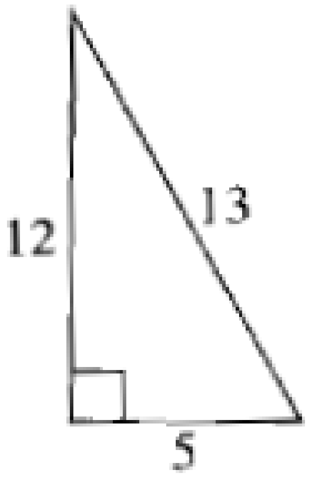


$6^2 + 8^2 > 9^2$

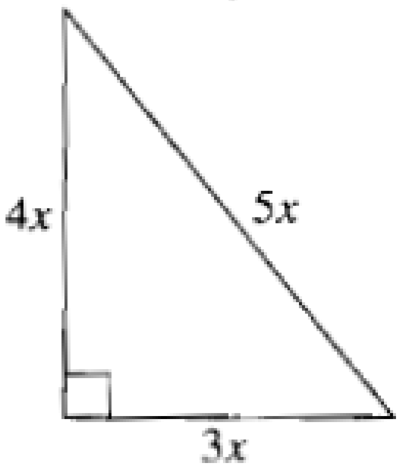
3, 4, 5 and 5, 12, 13 are Pythagorean triples, as well as 3k, 4k, 5k and 5k, 12k, 13k for any positive integer k. (they are the measures of the sides of right triangles)



3, 4, 5



5, 12, 13

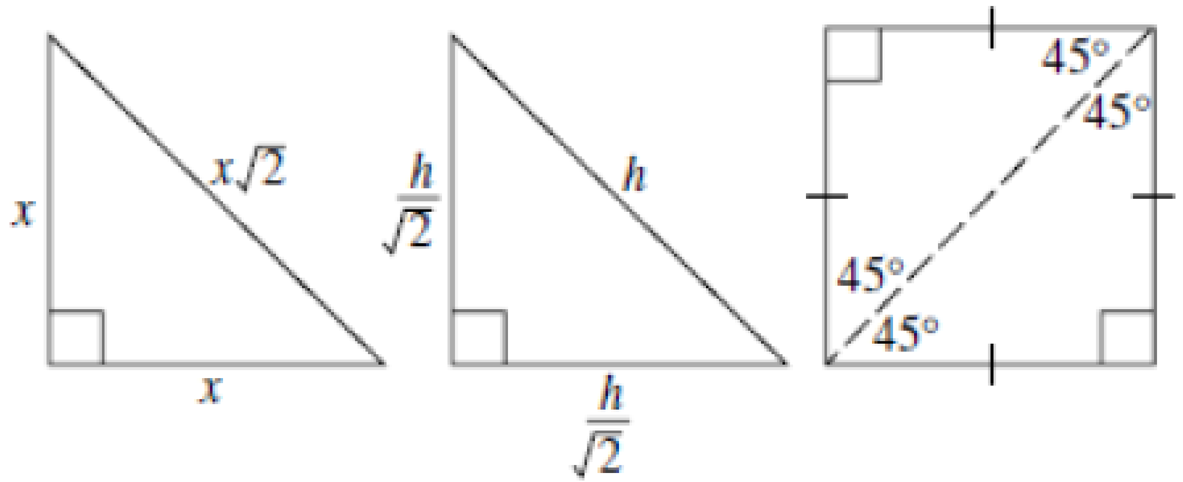


3x, 4x, 5x

The diagonal of a square divides the square into two isosceles right triangles.

In a 45-45-90 right triangle, the sides are x, x, and $x\sqrt{2}$. Therefore:

- **By multiplying the length of a leg by $\sqrt{2}$, you get the hypotenuse.**
- **By dividing the hypotenuse by $\sqrt{2}$, you get the length of each leg.**

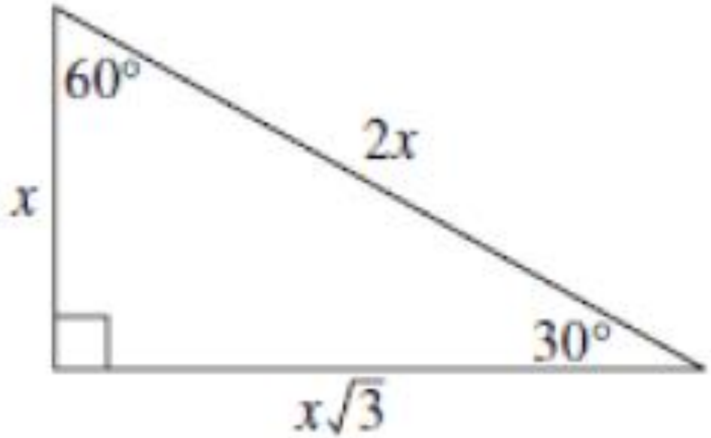


An altitude divides an equilateral triangle into two 30-60-90 right triangles.

In a 30-60-90 right triangle the sides are x, $x\sqrt{3}$, and $2x$.

If you know the length of the shorter leg (x):

- multiply it by $\sqrt{3}$ to get the length of the longer leg;
- multiply it by 2 to get the length of the hypotenuse.



Example 6.

What is the area of a square whose diagonal is 10?

(A) 20 (B) 40 (C) 50 (D) 100 (E) 200

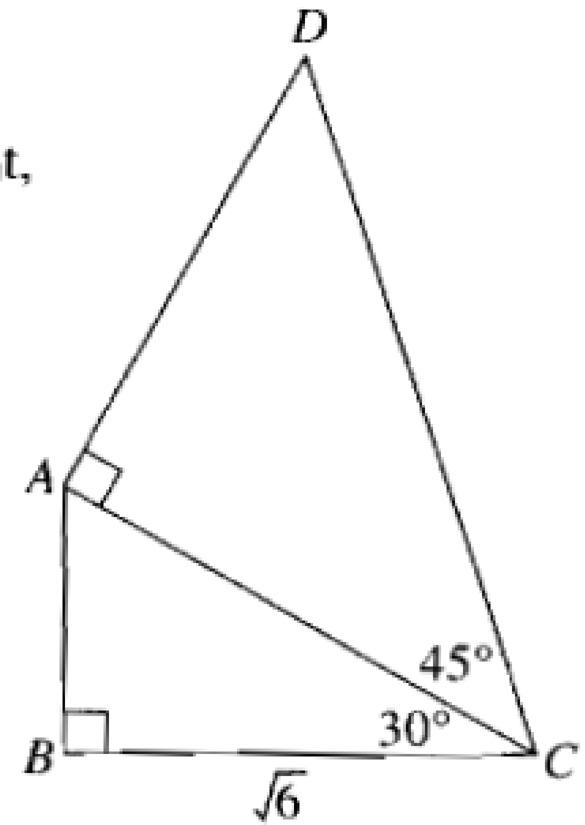
Example 7.

In the diagram at the right, if $BC = \sqrt{6}$, what is the value of CD?

(A) $2\sqrt{2}$ (B) $4\sqrt{2}$

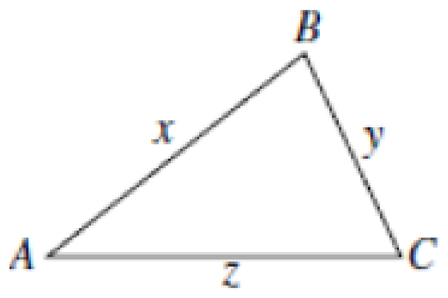
(C) $2\sqrt{3}$ (D) $2\sqrt{6}$

(E) 4



The Triangle Inequality Theorem

$$|x - y| < z < x + y$$
$$|z - y| < x < z + y$$
$$|x - z| < y < x + z$$



Example 8.

If the lengths of two of the sides of a triangle are 6 and 7, which of the following could be the length of the third side?

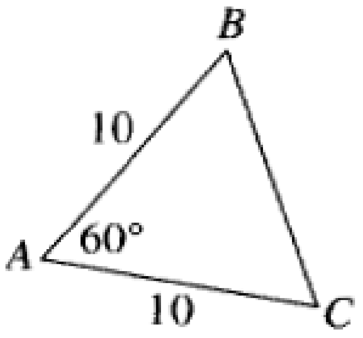
I. 1
II. 5
III. 15

(A) None (B) I only (C) II only (D) I and II only (E) I, II, and III

Example 9.

In the figure at the right, what is the perimeter of $\triangle ABC$?

(A) $20 + 10\sqrt{2}$
(B) $20 + 10\sqrt{3}$
(C) 25
(D) 30
(E) 40



Example 10.

What is the area of an equilateral triangle whose sides are 10?

(A) 30 (B) $25\sqrt{3}$ (C) 50 (D) $50\sqrt{3}$ (E) 100

K. QUADRILATERALS AND OTHER POLYGONS

A polygon is a closed figure, with straight sides that meet only at the vertices.

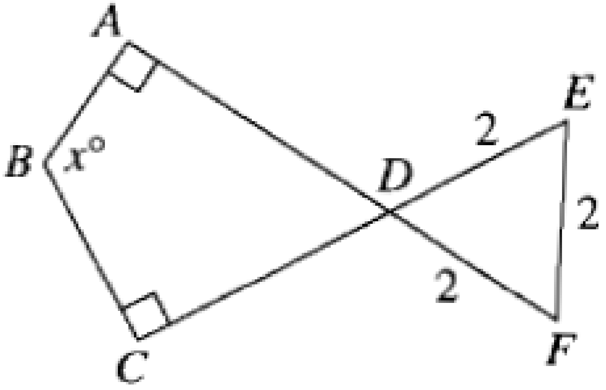
Number of sides	**Name**
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon

The sum of the measures of the n angles in a polygon with n sides is $(n - 2) \times 180°$.

In any polygon, the sum of the measures of the exterior angles, taking one at each vertex, is 360°.

Example 1.

In the figure below, what is the value of x?

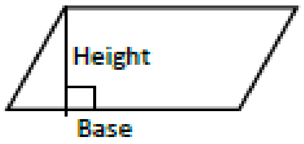
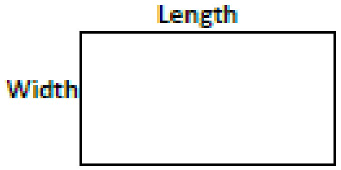
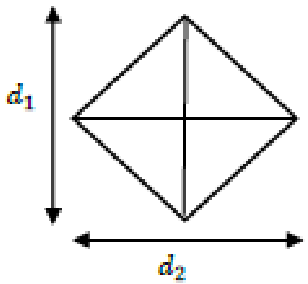
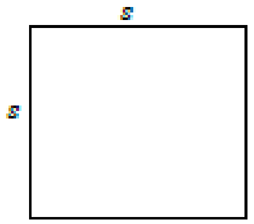


(A) 60 (B) 90 (C) 100 (D) 120 (E) 150

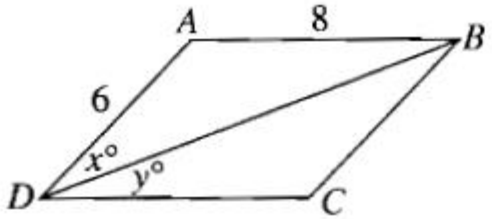
Example 2.

What is the measure of each interior angle in a regular decagon?

(A) 36 (B) 72 (C) 108 (D) 144 (E) 180

Shape	Area	Properties
Parallelogram 	**Area = Base × Height**	Opposite sides are parallel and congruent Opposite angles are congruent Adjacent angles are supplementary Diagonals bisect each other
Rectangle 	**Area = Length × Width**	All properties of a parallelogram All angles are right Diagonals are congruent and bisect each other
Rhombus 	**Area** $= \frac{1}{2} d_1 \times d_2$	- All properties of a parallelogram All sides are congruent Diagonals are perpendicular, they bisect each other and they are the angle bisectors
Square 	**Area** $= s^2$	- All properties of a parallelogram All sides are congruent All angles are right Diagonals are congruent, perpendicular, they bisect each other and they are the angle bisectors

Example 3.



ABCD is a parallelogram.

x	y

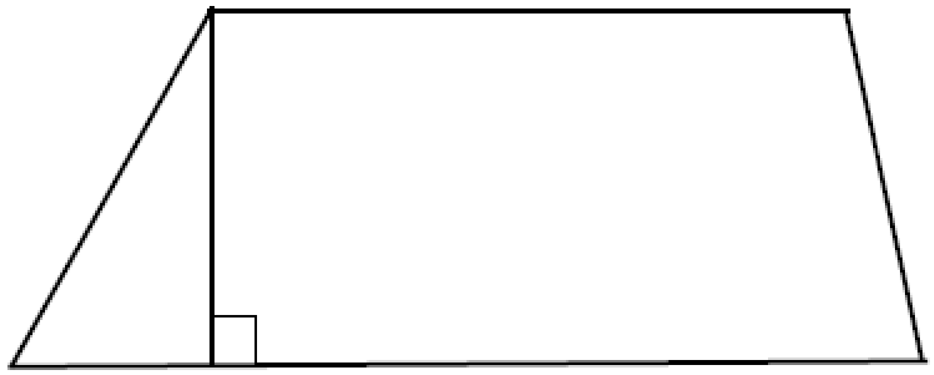
Example 4.

What is the length of each side of a square if its diagonals are 10?

(A) 5 (B) 7 (C) $5\sqrt{2}$ (D) $10\sqrt{2}$ (E) $10\sqrt{3}$

Trapezoid: A trapezoid has exactly one pair of parallel sides and 1 pair of non-parallel sides. The parallel sides are called the bases and the non-parallel sides are called the legs. The angles adjacent to the same leg are supplementary.

$$Area = \frac{Small\ base + Large\ base}{2} \times Height$$



Isosceles trapezoid: A trapezoid that has congruent legs is isosceles.
In an isosceles trapezoid, base angles are equal.

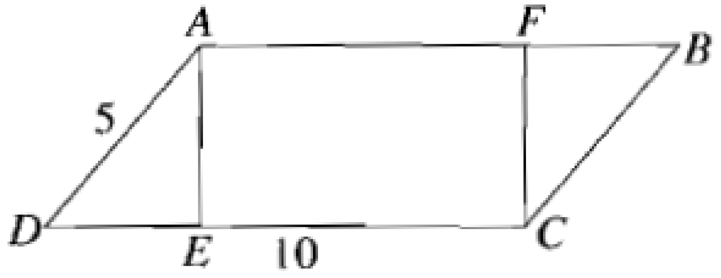
Example 5.

The length of a rectangle is 7 more than its width. If the perimeter of the rectangle is the same as the perimeter of a square of side 8.5, what is the length of a diagonal of the rectangle?

(A) 12 (B) 13 (C) 17 (D) 34 (E) 169

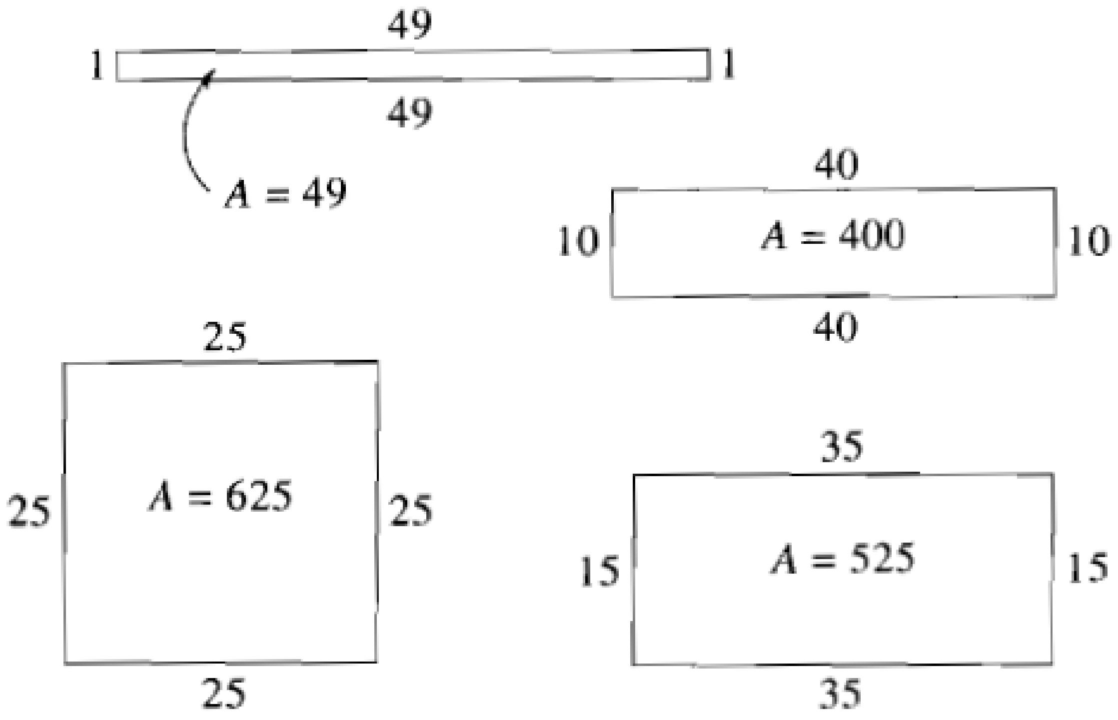
Example 6.

In the figure below, the area of parallelogram *ABCD* is 40. What is the area of rectangle *AFCE*?

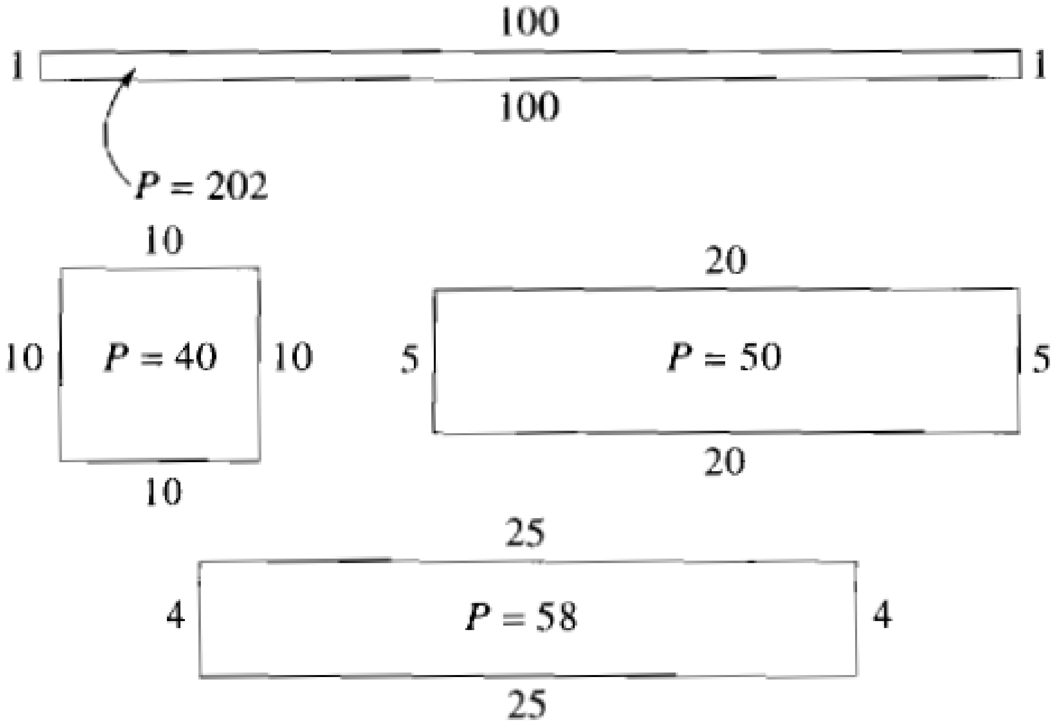


(A) 20 (B) 24 (C) 28 (D) 32 (E) 36

RECTANGLES WHOSE PERIMETERS ARE 100



RECTANGLES WHOSE AREAS ARE 100



For a given perimeter, the rectangle with the largest area is a square. For a given area, the rectangle with the smallest perimeter is a square.

Column A	Column B

Example 7.

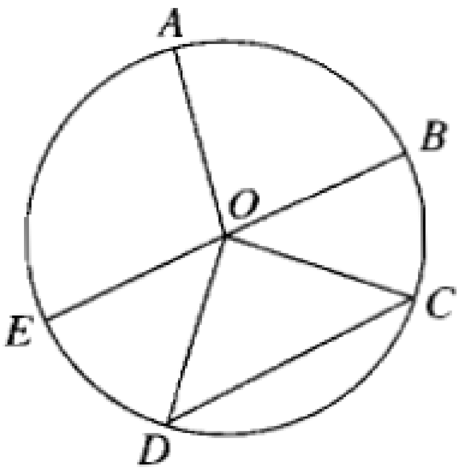
Column A	Column B
The area of a rectangle whose perimeter is 12	The area of a rectangle whose perimeter is 14

Example 8.

Column A	Column B
The area of a rectangle whose perimeter is 12	10

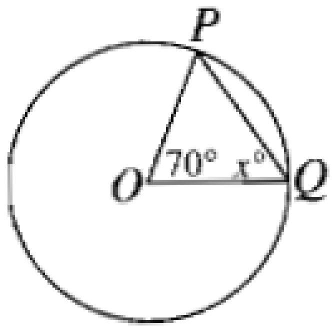
L. CIRCLES

A ***circle*** consists of all the points that are the same distance from one fixed point called the ***center***. That distance is called the ***radius*** of the circle. The figure below is a circle of radius 1 unit whose center is at the point *O*. *A*, *B*, *C*, *D*, and *E*, which are each 1 unit from *O*, are all points on circle *O*. The word *radius* is also used to represent any of the line segments joining the center and a point on the circle. The plural of ***radius*** is ***radii***. In circle *O*, below, *OA*, *OB*, *OC*, *OD*, and *OE* are all radii. If a circle has radius *r*, each of the radii is *r* units long.



Example 1.

If *P* and *Q* are points on circle *O*, what is the value of *x*?

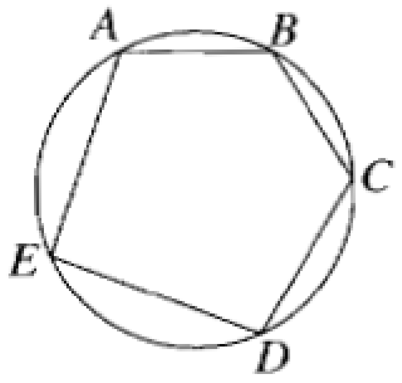


(A) 35 (B) 45 (C) 55 (D) 65 (E) 70

- $\pi = \dfrac{\text{circumference}}{\text{diameter}} = \dfrac{C}{d}$
- $C = \pi d$
- $C = 2\pi r$

Column A	Column B

Example 2.



The radius of the circle is 0.1.

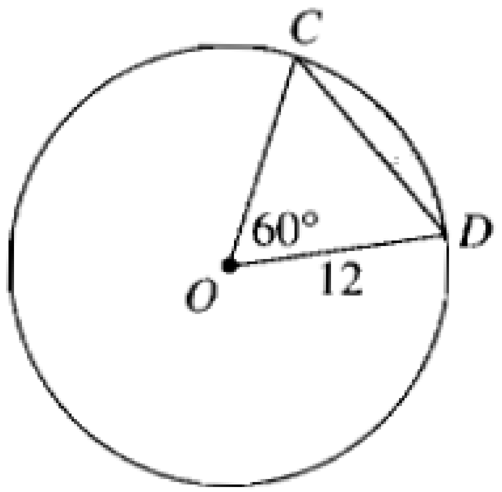
Column A	Column B
$AB + BC + CD + DE + EA$	1

Example 3.

Column A	Column B
The circumference of a circle whose diameter is 12	The perimeter of a square whose side is 12

If an arc measures $x°$, the length of the arc is $\frac{x}{360}(2\pi r)$, and the area of the sector formed by the arc and 2 radii is $\frac{x}{360}(\pi r^2)$.

Examples 4 and 5 refer to the circle below.



Example 4.

What is the area of the shaded region?

(A) $144\pi - 144\sqrt{3}$ (B) $144\pi - 36\sqrt{3}$ (C) $144 - 72\sqrt{3}$ (D) $24\pi - 36\sqrt{3}$ (E) $24\pi - 72\sqrt{3}$

Example 5.

What is the perimeter of the shaded region?

(A) $12 + 4\pi$ (B) $12 + 12\pi$ (C) $12 + 24\pi$ (D) $12\sqrt{2} + 4\pi$ (E) $12\sqrt{2} + 24\pi$

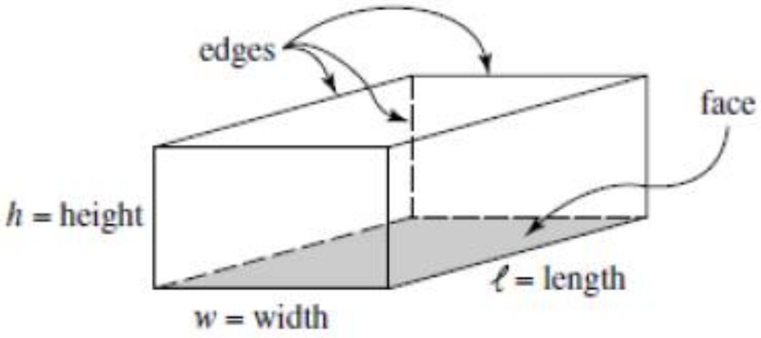
Example 6.

A is the center of a circle whose radius is 8, and *B* is the center of a circle whose diameter is 8. If these two circles are tangent to one another, what is the area of the circle whose diameter is *AB*?

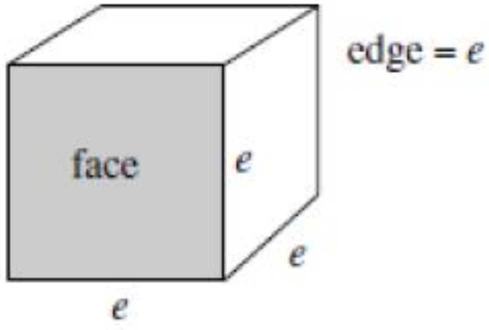
(A) 12π (B) 36π (C) 64π (D) 144π (E) 256π

M. SOLID GEOMETRY

Rectangular Solids and Cubes



RECTANGULAR SOLID



CUBE

The formula for the volume of a rectangular solid is $V = \ell wh$.

In a cube, all the edges are equal. Therefore, if e is the length of an edge, the formula for the volume is $V = e^3$.

The formula for the surface area of a rectangular solid is $A = 2(\ell w + \ell h + wh)$. The formula for the surface area of a cube is $A = 6e^2$.

Example 1.

The base of a rectangular tank is 12 feet long and 8 feet wide; the height of the tank is 30 inches. If water is pouring into the tank at the rate of 2 cubic feet per second, how many minutes will be required to fill the tank?

(A) 1 (B) 2 (C) 10 (D) 120 (E) 240

Example 2.

The volume of a cube is v cubic *yards*, and its surface area is a square *feet*. If $v = a$, what is the length in *inches* of each edge?

(A) 12 (B) 36 (C) 144 (D) 648 (E) 1944

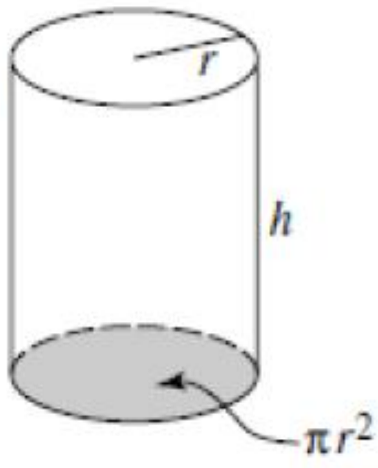
Example 3.

What is the length of a diagonal of a cube whose edges are 1?

(A) 1 (B) 2 (C) 3 (D) $\sqrt{2}$ (E) $\sqrt{3}$

Cylinders

A ***cylinder*** is similar to a rectangular solid except that the base is a circle instead of a rectangle. The volume of a cylinder is the area of its circular base (πr^2) times its height (h). The surface area of a cylinder depends on whether you are envisioning a tube, such as a straw, without a top or bottom, or a can, which has both a top and a bottom.



- **The volume, V, of a cylinder whose circular base has radius r and whose height is h is the area of the base times the height:**

$$V = \pi r^2 h.$$

- **The surface area, A, of the side of the cylinder is the circumference of the circular base times the height:**

$$A = 2\pi r h.$$

Column A	Column B

Example 4.

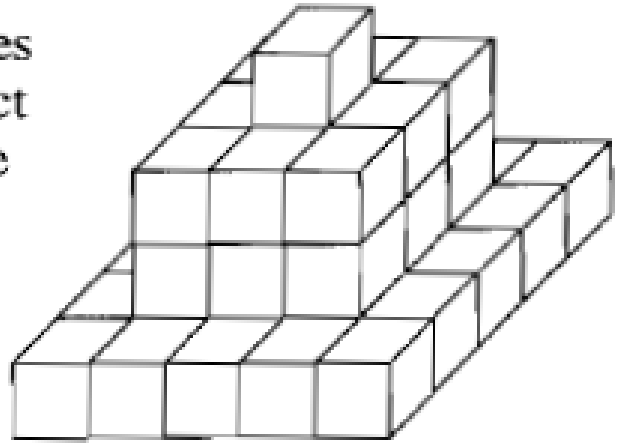
The radius of cylinder II equals the height of cylinder I.
The height of cylinder II equals the radius of cylinder I.

The volume of cylinder I	The volume of cylinder II

Example 5.

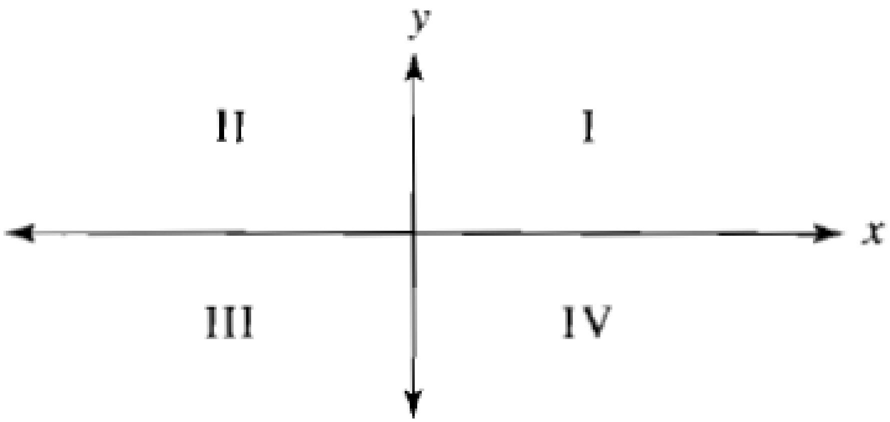
How many small cubes are needed to construct the tower in the figure at the right?

(A) 25 (B) 28 (C) 35 (D) 44 (E) 67



N. COORDINATE GEOMETRY

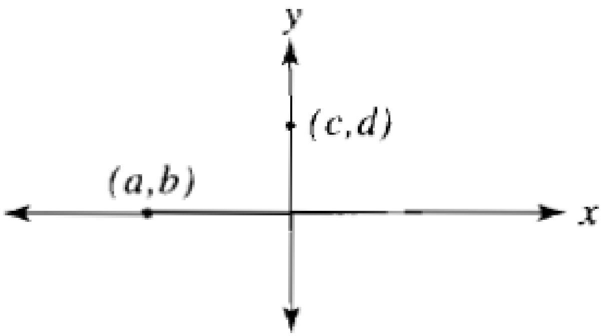
The coordinate plane is formed by two perpendicular number lines called the ***x-axis*** and ***y-axis***, which intersect at the ***origin***. The axes divide the plane into four ***quadrants***, labeled I, II, III, and IV.



Each point in the plane is assigned two numbers, an ***x-coordinate*** and a ***y-coordinate***, which are written as an ordered pair, (***x, y***).

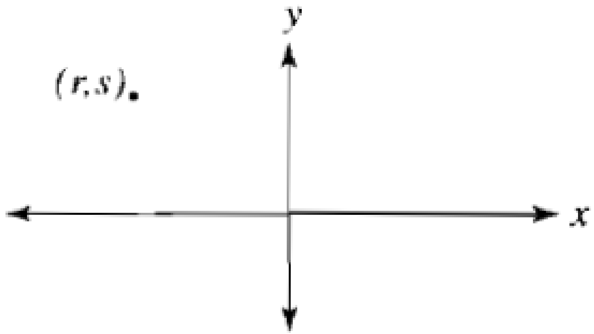
Column A | Column B

Example 1.



Column A	Column B
b	c

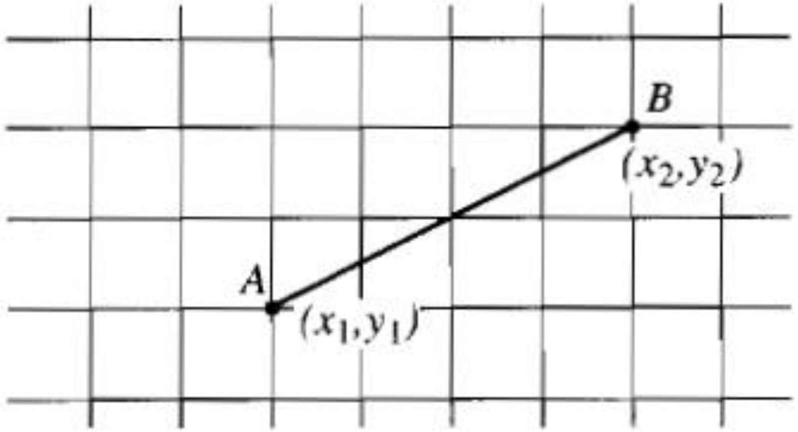
Example 2.



Column A	Column B
r	s

The distance, d, between two points, $A(x_1, y_1)$ and $B(x_2, y_2)$, can be calculated using the distance formula:

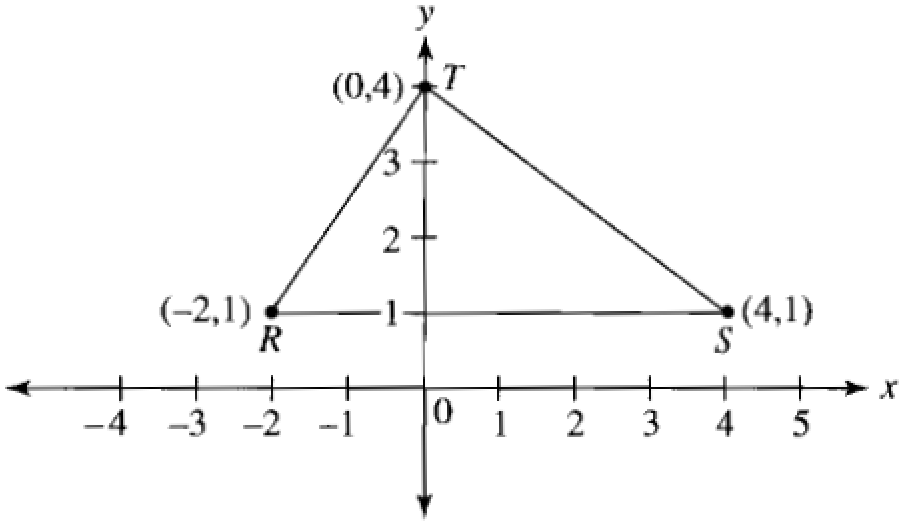
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



Example 3.

What is the area of $\triangle RST$?

(A) 6 (B) 9 (C) 12 (D) 15 (E) 18



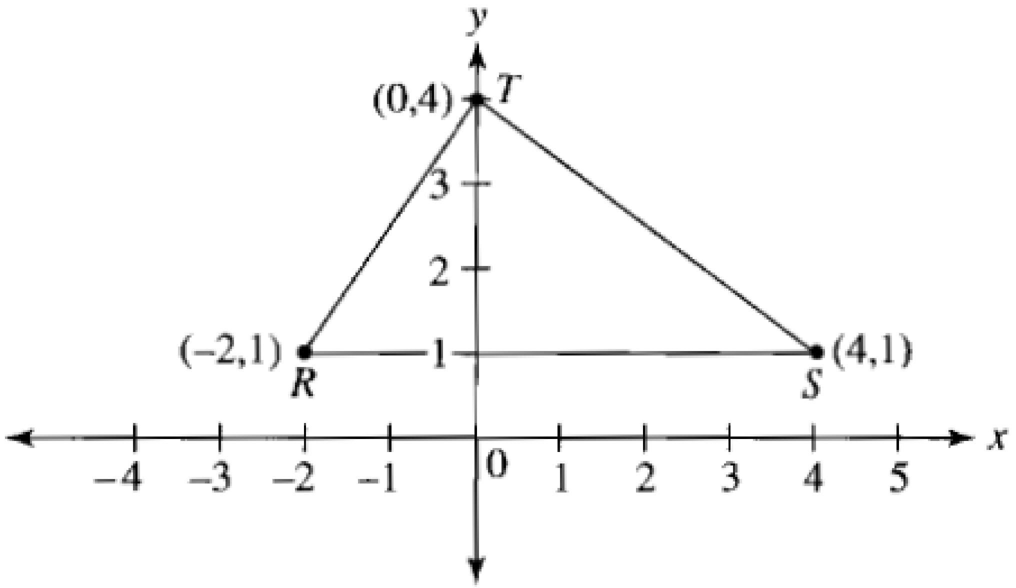
Example 4.

What is the perimeter of $\triangle RST$?

(A) 13 (B) 14 (C) 16 (D) $11 + \sqrt{13}$ (E) $11 + \sqrt{61}$

The ***slope*** of a line is a number that indicates how steep the line is.

- **To find the slope of any other line proceed as follows:**
  1. **Choose any two points A(x_1, y_1) and B(x_2, y_2) on the line.**
  2. **Take the differences of the y-coordinates, $y_2 - y_1$, and the x-coordinates, $x_2 - x_1$.**
  3. **Divide: slope = $\dfrac{y_2 - y_1}{x_2 - x_1}$.**



- **Vertical lines *do not have slopes.***

- **The slope of any horizontal line is 0:**

  slope of *RS* $= \dfrac{1-1}{4-(-2)} = \dfrac{0}{6} = 0$

- **The slope of any line that goes up as you move from left to right is positive: slope of *RT*** $= \dfrac{4-1}{0-(-2)} = \dfrac{3}{2}$

- **The slope of any line that goes down as you move from left to right is negative: slope of *ST*** $= \dfrac{1-4}{4-0} = \dfrac{-3}{4} = -\dfrac{3}{4}$

Column A	Column B

Example 5.

Line ℓ passes through (1, 2) and (3, 5)
Line m is perpendicular to ℓ

The slope of ℓ	The slope of m

0. COUNTING AND PROBABILITY

Example 1.

Brian bought some apples. If he entered the store with $113 and left with $109, how much did the apples cost?

Example 2.

Scott was selling tickets for the school play. One day he sold tickets numbered 109 through 113. How many tickets did he sell that day?

Example 3.

Brian is the 109th person in a line and Scott is the 113th person. How many people are there between Brian and Scott?

To count how many integers there are between two integers, follow these rules:

- **If exactly one of the endpoints is included, subtract.**
- **If both endpoints are included, subtract and add 1.**
- **If neither endpoint is included, subtract and subtract 1 more.**

Example 4.

From 1:09 to 1:13, Adam read pages 109 through 113 in his English book. What was his rate of reading, in pages per minute?

(A) $\frac{3}{5}$ (B) $\frac{3}{4}$ (C) $\frac{4}{5}$ (D) 1 (E) $\frac{5}{4}$

If two jobs need to be completed and there are *m* ways to do the first job and *n* ways to do the second job, then there are $m \times n$ ways to do one job followed by the other. This principle can be extended to any number of jobs.

Example 5.

Ariel has 4 paintings in the basement. She is going to bring up 2 of them and hang 1 in her den and 1 in her bedroom. In how many ways can she choose which paintings go in each room?

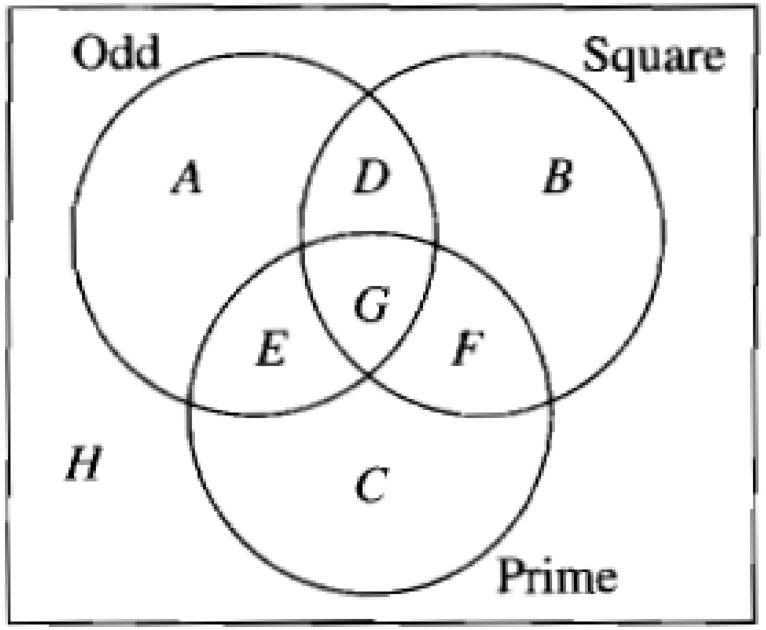
(A) 4 (B) 6 (C) 12 (D) 16 (E) 24

Example 6.

How many integers are there between 100 and 1000 all of whose digits are odd?

Example 7.

If the integers from 1 through 15 are each placed in the diagram at the right, which regions are empty?



(A) D only (B) F only (C) G only (D) F and G only (E) D and G only

Example 8.

Of the 410 students at H. S. Truman High School, 240 study Spanish and 180 study French. If 25 students study neither language, how many study both?

(A) 25 (B) 35 (C) 60 (D) 170 (E) 230

If E is any event, the probability that E will occur is given by $$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}},$$ **assuming that the possible outcomes are all equally likely.**

Let E be an event, and $P(E)$ the probability it will occur.

- If E is ***impossible*** (such as getting a number greater than 10), $P(E) = 0$.
- If it is ***certain*** that E will occur (such as getting a prime number), $P(E) = 1$.
- In all cases $0 \le P(E) \le 1$.
- The probability that event E will not occur is $1 - P(E)$.

Example 9.

An integer between 100 and 999, inclusive, is chosen at random. What is the probability that all the digits of the number are odd?

Example 10.

A fair coin is flipped three times. What is the probability that the coin lands heads each time?

Column A	Column B

Example 11.

Three fair coins are flipped.

The probability of getting more heads than tails	The probability of getting more tails than heads

Column A	Column B

Example 12.

The numbers from 1 to 1000 are each written on a slip of paper and placed in a box. Then 1 slip is removed.

The probability that the number drawn is a multiple of 5	The probability that the number drawn is a multiple of 7